The closest normal structured matrix

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OUTLINE

- Introduction
- Structured matrices
  and structure-preserving transformations
- Jacobi-type algorithm
  for the reduction to the canonical form
- Finding the closest normal matrix with a given structure
- Numerical examples

E. Begović Kovač, H. Faßbender, P. Saltenberger: On normal and structured matrices under unitary structure-preserving transformations. arXiv:1810.03369 [math.NA]

INTRODUCTION

- Set of normal matrices: $\mathcal{N} = \{X \ : \ XX^H = X^H X\}$
- X is normal if and only if there is unitary U such that

$$U^H X U = \begin{bmatrix} \diagdown \end{bmatrix}.$$

- A. Ruhe: *Closest normal matrix finally found!*
  BIT 27 (4) (1987) 585–598.

  Does NOT preserve given matrix structure.

INTRODUCTION

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Suppose that A has a structure $\mathcal{S}$, $A \in \mathcal{S}$.

Minimization problem:

$$\min \left\{ \|A - X\|_F^2 \; : \; X \in \mathcal{N} \cap \mathcal{S} \right\}$$

MAXIMIZATION PROBLEM

Theorem (Causey 1964, Gabriel 1979)

Let $A \in \mathbb{C}^{n \times n}$ and let $X = ZDZ^H$, where Z is unitary and D is diagonal. Then X is a nearest normal matrix to A in the Frobenius norm if and only if

(a) $\|\mathrm{diag}(Z^H A Z)\|_F = \max_{QQ^H = I} \|\mathrm{diag}(Q^H A Q)\|_F$, and

(b) $D = \mathrm{diag}(Z^H A Z)$.

→ Finding the closest normal matrix is equivalent to finding an unitary transformation that maximizes Frobenius norm of the diagonal.

→ This theorem has to be modified to fulfill structure-preserving requirements.

- N. J. Higham: *Matrix nearness problem and applications*. In Applications of Matrix theory 22 (1989) 1–27.

STRUCTURED MATRICES

- **Hamiltonian** A (J-Hermitian):

$$(JA)^H = JA, \quad \text{that is } A^H = JAJ, \quad \text{where} \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

We can write it as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & -A_{11}^H \end{bmatrix}, \qquad A_{12}^H = A_{12},\ A_{21}^H = A_{21}.$$

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- **Skew-Hamiltonian** A (J-skew-Hermitian):

$$(JA)^H = -JA, \quad \text{that is } A^H = -JAJ.$$

We can write it as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{11}^H \end{bmatrix}, \qquad A_{12}^H = -A_{12},\ A_{21}^H = -A_{21}.$$

- For every skew-Hamiltonian W there is Hamiltonian H (and viceversa) such that $W = \imath H$.

STRUCTURED MATRICES–cont.

- **Per-Hermitian** A (F-Hermitian):

$$(FA)^H = FA, \quad \text{that is } A^H = FAF,$$

$$\text{where} \quad F = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \iddots & 0 \\ 0 & \iddots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

$\rightarrow$ Hermitian about its anti-diagonal

- **Perskew-Hermitian** A (F-skew-Hermitian):

$$(FA)^H = -FA, \quad \text{that is } A^H = -FAF.$$

$\rightarrow$ Skew-Hermitian about its anti-diagonal

- For every perskew-Hermitian K there is per-Hermitian M (and viceversa) such that $K = \imath M$.

STRUCTURE-PRESERVING TRANSFORMATIONS

- For Hamiltonian and skew-Hamiltonian

$$M \text{ is } \textbf{symplectic} \text{ if } M^H J M = J.$$

- For per-Hermitian and perskew-Hermitian

$$M \text{ is } \textbf{perplectic} \text{ if } M^H F M = F.$$

CANONICAL FORM — HAMILTONIAN

Theorem (BK, Faßbender, Saltenberger)

For every normal Hamiltonian $A \in \mathbb{C}^{2n\times 2n}$ there is unitary symplectic U such that

$$U^H A U = \begin{bmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_2 & 0 & D_3 \\ 0 & 0 & -D_1^H & 0 \\ 0 & -D_3 & 0 & D_2 \end{bmatrix},$$

where D_j, $j = 1, 2, 3$ diagonal matrices,
$D_1 \in \mathbb{C}^{n_1\times n_1}$, $D_2 \in \imath\mathbb{R}^{n_2\times n_2}$, $D_3 \in \mathbb{R}^{n_2\times n_2}$, $n_1 + n_2 = n$.

$$U^H A U = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ -\Lambda_2 & -\Lambda_1^H \end{bmatrix} = \begin{bmatrix} \diagdown & \diagdown \\ \diagdown & \diagdown \end{bmatrix} =: \Lambda_{\mathcal{H}}$$

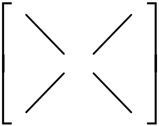
Theorem (BK, Faßbender, Saltenberger)

For every normal per-Hermitian $A \in \mathbb{C}^{2n \times 2n}$ there is unitary perplectic U such that

$$U^H A U = \begin{bmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_2 & D_3 & 0 \\ 0 & FD_3F & FD_2F & 0 \\ 0 & 0 & 0 & FD_1F \end{bmatrix},$$

where D_1 i D_2 are diagonal, and D_3 is antidiagonal matrix, $D_1 \in \mathbb{C}^{n_1 \times n_1}$, $D_2 \in \mathbb{R}^{n_2 \times n_2}$, $D_3 \in \mathbb{R}^{n_2 \times n_2}$, $n_1 + n_2 = n$.

$$U^H A U = \begin{bmatrix} \Lambda_1 & \Lambda_2 F \\ F\Lambda_2 & F\Lambda_1^H F \end{bmatrix} =$$



$$=: \Lambda_{\mathcal{P}}$$

MAXIMIZATION ALGORITHM

$$\max_{ZZ^H=I,\ Z\in Sp_{2n}(\mathbb{C})} \left\{ f_{\mathcal{H}}(Z) := \|\mathrm{diag}(Z^H A Z)\|_F^2 + \|\mathrm{diag}(J Z^H A Z)\|_F^2 \right\}$$

- Iterative algorithm of the form

$$A^{(k+1)} = R_k^H A^{(k)} R_k, \quad k \geq 0.$$

- Transformations R_k are structure-preserving rotations obtained by embedding **two Jacobi rotations**

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} := \begin{bmatrix} \cos\phi & -e^{\imath\alpha}\sin\phi \\ e^{-\imath\alpha}\sin\phi & \cos\phi \end{bmatrix} \qquad \text{in } I_{2n}.$$

  They are chosen to maximize

$$\|\mathrm{diag}(A^{(k+1)})\|_F^2 + \|\mathrm{diag}(JA^{(k+1)})\|_F^2.$$

- D. S. Mackey, N. Mackey, F. Tisseur: *Structured tools for structured matrices*. Electron. J. Linear Al. 10 (2003) 106–145.

MAXIMIZATION ALGORITHM

$$\max_{ZZ^H=I,\ Z\in Pp_{2n}(\mathbb{C})} \left\{ f_{\mathcal{P}}(Z) := \|\mathrm{diag}(Z^H A Z)\|_F^2 + \|\mathrm{diag}(F Z^H A Z)\|_F^2 \right\}$$

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SYMPLECTIC ROTATIONS

$$R(i,j,\phi,\alpha) = \left[\begin{array}{cc|cc} c & -s & & \\ \bar{s} & c & & \\ \hline & & c & -s \\ & & \bar{s} & c \end{array}\right] \begin{array}{c} i \\ j \\ \\ n+i \\ n+j \end{array}$$

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PIVOT POSITIONS (SYMPLECTIC)

$$
\left[\begin{array}{cccc|cccc}
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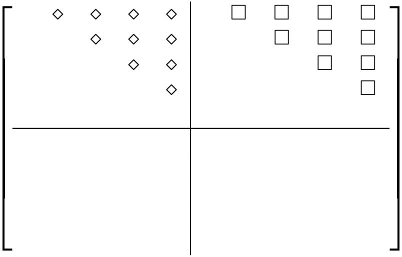
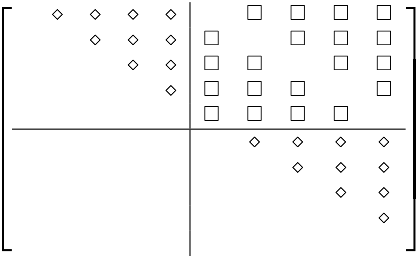


considering double rotations
$\rightsquigarrow$



PERPLECTIC ROTATIONS

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$$R(i,j,\phi,\alpha) = \left[\begin{array}{cc|cc} c & & -s & \\ & c & & \bar{s} \\ \hline \bar{s} & & c & \\ & -s & & c \end{array}\right] \begin{array}{c} i \\ 2n-j+1 \\ j \\ 2n-i+1 \end{array}$$

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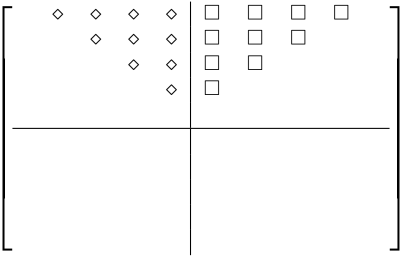

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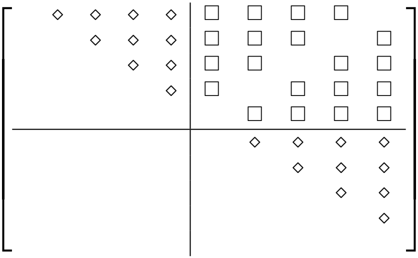

REDUCTION TO CANONICAL FORM

Jacobi-type algorithm 1

Input: $A \in \mathbb{C}^{2n \times 2n} \in \mathcal{S}$, $Z_0 = I$
Output: structure-preserving unitary Z
REPEAT
  Select (i_k, j_k).
  Find ϕ_k and α_k for $R(i_k, j_k, \phi_k, \alpha_k)$.
  $A^{(k+1)} = R_k^H A^{(k)} R_k$
  $Z_{k+1} = Z_k R_k$
UNTIL convergence

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- Cyclic pivot strategy
- Convergence condition:

$$|\langle \text{grad} f(Z), Z\dot{R}(i_k, j_k, 0, \alpha_k) \rangle| \geq \eta \|\text{grad} f(Z)\|_F,$$

where $\dot{R}(i,j,\phi,\alpha) = \frac{\partial}{\partial \phi} R(i,j,\phi,\alpha)$ and $f = f_{\mathcal{H}}$ or $f = f_{\mathcal{P}}$.

CONVERGENCE

Theorem (BK, Faßbender, Saltenberger)

Let A be Hamiltonian (or skew-Hamiltonian) and let $(Z_k)_k$ be a sequence of unitary symplectic matrices generated by the Jacobi algorithm. Every accumulation point of $(Z_k)_k$ is a stationary point of function $f_{\mathcal{H}}$.

Theorem (BK, Faßbender, Saltenberger)

Let A be per-Hermitian (or perskew-Hermitian) and let $(Z_k)_k$ be a sequence of unitary perplectic matrices generated by the Jacobi algorithm. Every accumulation point of $(Z_k)_k$ is a stationary point of function $f_{\mathcal{P}}$.

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- M. Ishteva, P.-A. Absil, P. Van Dooren: *Jacobi algorithm for the best low multilinear rank approximation of symmetric tensors.*
  SIAM J. Matrix Anal. Appl. 34(2) (2013) 651–672.

THE CLOSEST NORMAL MATRIX

- Let A be Hamiltonian. Analogy with unstructured case:
  1. (i) Find Z that maximizes
     $f_{\mathcal{H}}(Z) = \|\mathrm{diag}(Z^H A Z)\|_F^2 + \|\mathrm{diag}(J Z^H A Z)\|_F^2$,
  2. (ii) Extract the canonical form,
  3. (iii) Solution is given by $X = Z \begin{bmatrix} \diagdown & \diagdown \\ \diagdown & \diagdown \end{bmatrix} Z^H$.

$\rightarrow$ But this can produce a matrix that is not normal!

THE CLOSEST NORMAL MATRIX

- Let A be Hamiltonian. Analogy with unstructured case:
  - (i) Find Z that maximizes
    $f_{\mathcal{H}}(Z) = \|\mathrm{diag}(Z^H AZ)\|_F^2 + \|\mathrm{diag}(JZ^H AZ)\|_F^2$,
  - (ii) Extract the canonical form,
  - (iii) Solution is given by $X = Z \begin{bmatrix} \ddots & \ddots \\ \ddots & \ddots \end{bmatrix} Z^H$.

  $\rightarrow$ But this can produce a matrix that is not normal!

- We set

$$f_{\mathcal{D}}(Z) = \|\mathrm{diag}(Z^H AZ)\|_F^2.$$

  - (i) Find Z that maximizes $f_{\mathcal{D}}$.
  - (ii) Extract the diagonal.
  - (iii) Solution is given by $X = Z \begin{bmatrix} \ddots & \\ & \ddots \end{bmatrix} Z^H$.

ADDITIONAL ROTATIONS

$\rightarrow$ To find Z that maximizes $f_{\mathcal{D}}$ we add new rotations to the Jacobi algorithm.

- Symplectic rotations

$$R(i, n+i, \phi, 0) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{matrix} i \\ n+i \end{matrix}$$

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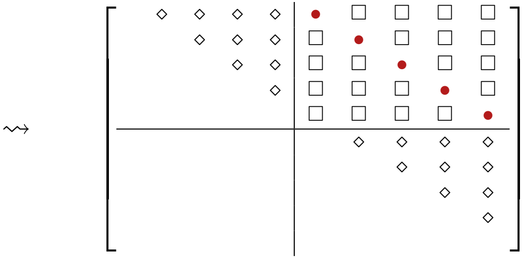

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- Perplectic rotations

$$R(i, 2n-i+1, \phi, -\frac{\pi}{2}) = \begin{bmatrix} \cos\phi & \imath\sin\phi \\ \imath\sin\phi & \cos\phi \end{bmatrix} \begin{matrix} i \\ 2n-i+1 \end{matrix}$$

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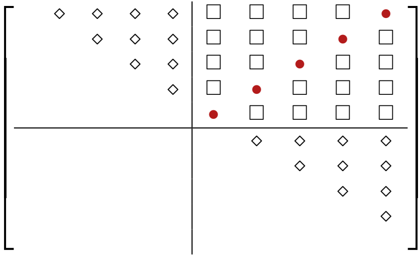

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DIAGONALIZATION ALGORITHM

Jacobi-type algorithm 2

Input: $A \in \mathbb{C}^{2n \times 2n} \in \mathcal{S}$, $Z_0 = I$
Output: structure-preserving unitary Z
REPEAT
    Select (i_k, j_k). (additional pivot positions are included)
    Find ϕ_k and α_k for $R(i_k, j_k, \phi_k, \alpha_k)$.
    $A^{(k+1)} = R_k^H A^{(k)} R_k$
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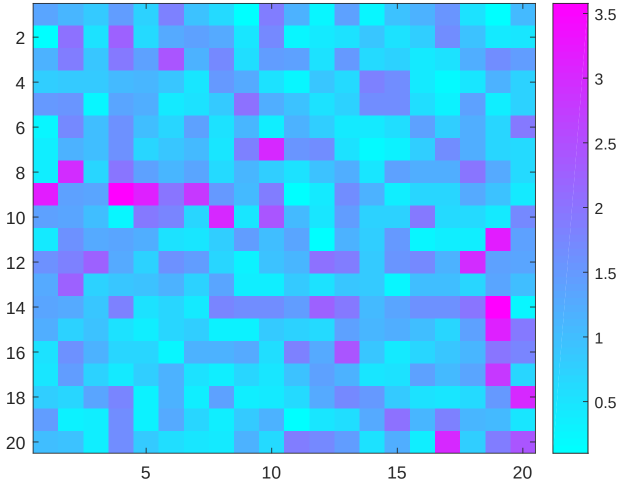
Theorem (BK, Faßbender, Saltenberger)

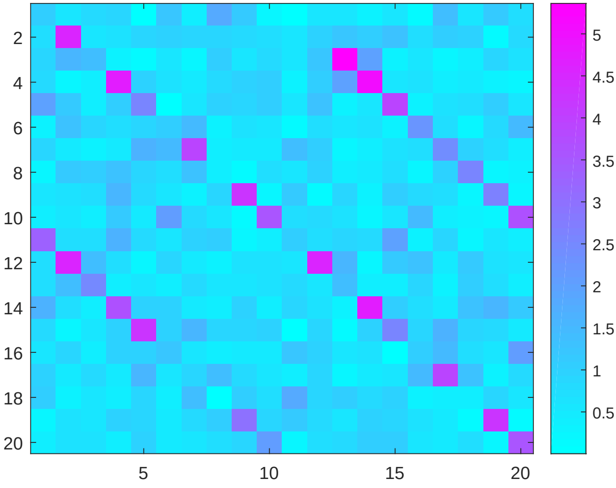
Let A be Hamiltonian and let $(Z_k)_k$ be a sequence of unitary symplectic matrices generated by the Jacobi algorithm with additional rotations. Every accumulation point of $(Z_k)_k$ is a stationary point of function $f_{\mathcal{D}}$.

NUMERICAL EXAMPLES — Canonical form



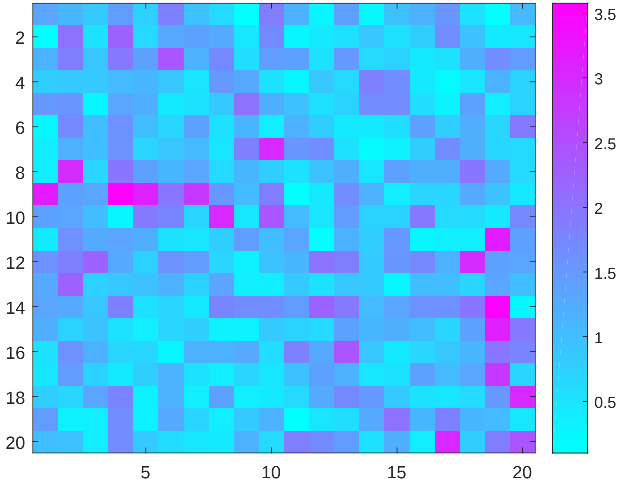
Random Hamiltonian 20×20 matrix.

NUMERICAL EXAMPLES — Canonical form



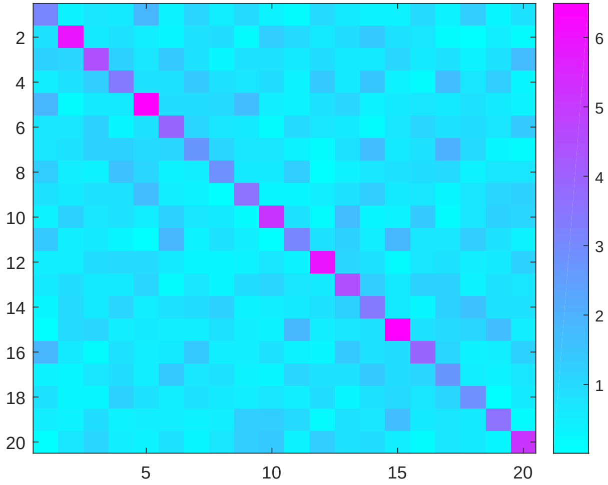
Reduction to the canonical form (Algorithm 1) after 10 cycles.

NUMERICAL EXAMPLES — Diagonalization



The same Hamiltonian 20×20 matrix.

NUMERICAL EXAMPLES — Diagonalization

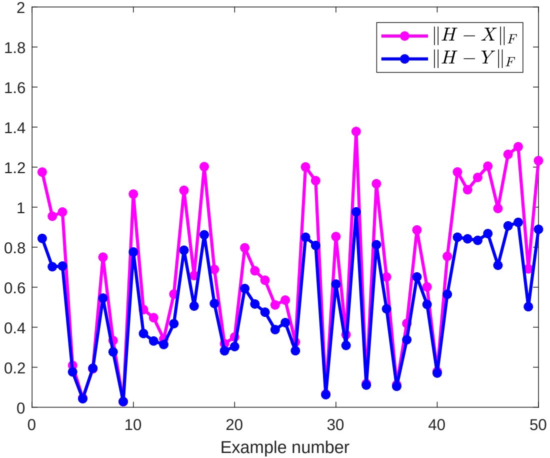


Diagonalization (Algorithm 2) after 10 cycles.

NUMERICAL EXAMPLES — Distance from normal matrix

We take normal Hamiltonian X and set $H = X + E$, such that H is Hamiltonian, but not normal.

Algorithm 2 on H gives its closest normal Y.



NUMERICAL EXAMPLES — Departure from normality

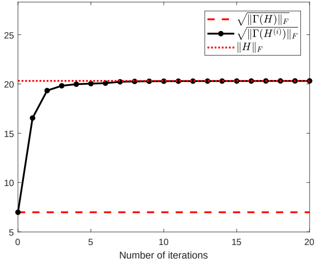
For any matrix A its Schur form

$$U^H A U = T = D + N$$

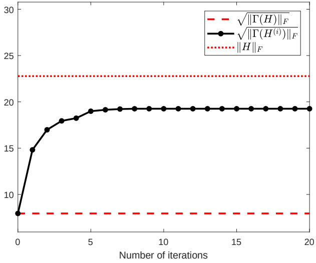
exists, where U is unitary, $D = \mathrm{diag}(T)$ and N is strictly upper triangular. The quantity $\Delta(A) = \|N\|_F$ is referred to as A's departure from normality.

We compare $\Delta(H)$ and $\mathrm{off}(H^{(20)})$ where $H^{(20)}$ is obtained by 20 iterations of Algorithm 2 and $\mathrm{off}(A) = \|A - \mathrm{diag}(A)\|_F^2$.

Example i	Size of H_i	$\Delta(H_i)$	$\mathrm{off}(H_i^{(20)})$
1	10	$7.1 \cdot 10^{+0}$	$6.4 \cdot 10^{+0}$
2	10	$4.0 \cdot 10^{-3}$	$3.1 \cdot 10^{-3}$
3	20	$3.5 \cdot 10^{-5}$	$3.1 \cdot 10^{-5}$
4	20	$5.3 \cdot 10^{+2}$	$4.4 \cdot 10^{+2}$
5	30	$7.7 \cdot 10^{+0}$	$6.7 \cdot 10^{+0}$
6	30	$1.0 \cdot 10^{-1}$	$9.0 \cdot 10^{-2}$
7	40	$7.9 \cdot 10^{-7}$	$6.6 \cdot 10^{-7}$
8	40	$3.1 \cdot 10^{+3}$	$2.7 \cdot 10^{+3}$
9	50	$1.1 \cdot 10^{-2}$	$9.5 \cdot 10^{-3}$
10	100	$7.8 \cdot 10^{-7}$	$6.8 \cdot 10^{-7}$



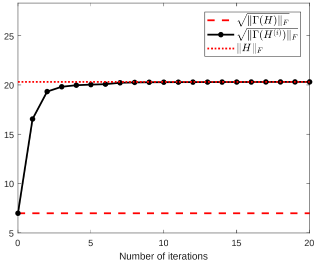
Normal Hamiltonian 20×20



Random Hamiltonian 20×20

$$\Gamma(A) := \|\mathrm{diag}(Z^H A Z)\|_F^2 + \|\mathrm{diag}(J Z^H A Z)\|_F^2$$

NUMERICAL EXAMPLES — Convergence of Algorithm 1



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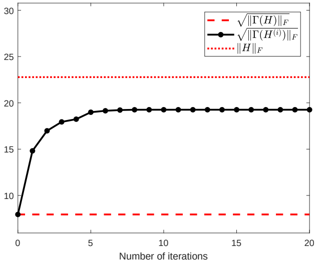

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THANK YOU!